(1) Note that $\forall a \in A$, we have $aB \subset A.B$
But since $|A.B| = |B| = |aB|$, it follows
$A.B = aB$, $\forall a \in A$.

In particular, $\forall a_1, a_2 \in A$, we have $a_1^{-1} a_2 B = B$
$\Rightarrow a_1^{-1} a_2 \in H = \mathrm{Stab}(B)$
Fix $g_0 \in A$. Then $a \in g_0 H$, $\forall a \in A$.

Now note that if $b \in B$, then $Hb \subset B$
(by definition of H).
Conclusion follows.

(2) a) Note that $(a,b) \in A \times B$ is such
that $a+b = x \iff a = x-b \in A \cap (x-B)$.

So $r(x) = |A \cap (x-B)|$.

b) $$E(A,B) = \left|\left\{ (a_1, a_2, b_1, b_2) \in A \times A \times B \times B : \begin{array}{c} \\ a_1 + b_1 = a_2 + b_2 \end{array} \right\}\right|$$

$$= |\{(a_1, a_2, b_1, b_2) \in A \times A \times B \times B : a_1 - a_2 = b_1 - b_2\}|$$

$$= \sum_{x \in (A-A) \cap (B-B)} |\{(a_1, a_2) \in A \times A : a_1 - a_2 = x\}| \cdot |(b_1, b_2) \in B \times B : b_2 - b_1 = x\}|$$

$$= \sum_{x \in (A-A) \cap (B-B)} |A \cap (x+A)| \, |B \cap (x+B)|.$$

③

a) We want to construct an injective map

$$f : \{(a,b) \in A \times B \mid ab = x_0\} \times B.A \longrightarrow B.A^{-1} \times B^{-1}.A.$$

For $x \in B.A$, choose $a(x) \in A$, $b(x) \in B$ s.t. $b(x)a(x) = x$.

We construct the map given by

$$f(a,b,x) = (b(x)a^{-1}, \; b^{-1}a(x)).$$

Note that if $f(a,b,x) = (u,v)$,

then $u x_0 v = x$,

Then we can recover $a(x)$ and $b(x)$.

Therefore choice of a, b, x is unique.

b) Note that $|\{(a,b) \in A \times B \mid ab = x_0\}|$

$$= |A \cap x_0 B^{-1}|.$$

Conclusion follows.

c) Note that if G abelian, then $A.B = B.A$,

$\forall A, B \subset G$.

Then $|B \cdot A^{-1}| = |(B \cdot A^{-1})^{-1}| = |A \cdot B^{-1}|$
$= |B^{-1} \cdot A|$.

④ (ii) $\Longleftrightarrow$ (iii) by definition of Ruzsa distance

(i) $\Rightarrow$ (ii) Apply previous exercise with $B = -A$, $x = 0$ (additive identity).

Then $$|A| \le \frac{|A+A|^2}{|A-A|}.$$

If $|A+A| \le K^{c_2}|A|$, then $|A-A| \le K^{c_2}|A|$.

(ii) $\Rightarrow$ (i). By Ruzsa inequality,
$$d(A,A) \le d(A,-A) + d(-A,A)$$
So $$\log\left(\frac{|A-A|}{|A|}\right) \le 2\log\left(\frac{|A+A|}{|A|}\right).$$

Conclusion follows.

(i) $\Rightarrow$ (iv) This is Pluennecke

(iv) $\Rightarrow$ (i) trivial

(v) $\Rightarrow$ (i) $|A+A| \le |H+H| \le K^{c_5}|H|$
$\le K^{2c_5}|A|$.

(i) $\Rightarrow$ (v) Take $H = A - A$.

Apply Ruzsa covering lemma to deduce

$$|3A-2A| \le K^c |A| \Rightarrow \exists X \subset 2A-2A,\ |X| \le K^c,$$
$$H+H = 2A-2A \subset A-A+X$$
$$= H+X.$$

So H is K^c-approximate group.

5 a) It suffice to show $|H_1 - H_2||H_1 \cap H_2| = |H_1| \cdot |H_2|$.

We construct bijection

$$f : (H_1 - H_2) \times (H_1 \cap H_2) \longrightarrow H_1 \times H_2$$

For $x \in H_1 - H_2$, choose $h_1(x) \in H_1$, $h_2(x) \in H_2$, s.t. $h_1(x) - h_2(x) = x$.

Define $f(x, h) = (h_1(x) + h,\ -h + h_2(x))$.

Injective: For $(u, v) \in f((H_1 - H_2) \times (H_1 \cap H_2))$, we have $u + v = x$.

Can recover $h_1(x)$ and $h_2(x)$, thus h. ✓

Surjective: Let $(h_1, h_2) \in H_1 \times H_2$.

Let $x = h_1 - h_2$.

Note that $h_1 - h_2 = h_1(x) - h_2(x)$.

Let $h = h_1 - h_1(x) = h_2 - h_2(x) \in H_1 \cap H_2$

then $f(x,h) = (h_1, h_2)$. ✓

b) Note that

$$\exp\big(d(H_1, H_1+H_2) + d(H_2, H_1+H_2)\big)$$

$$= \frac{|H_1+H_1+H_2|\,|H_2+H_1+H_2|}{\sqrt{|H_1|\,|H_1+H_2|^2\,|H_2|}}$$

$$= \frac{|H_1+H_2|}{\sqrt{|H_1||H_2|}} = \exp\big(d(H_1,H_2)\big)$$

(Since H_1, H_2 are subgroups).

For second inequality,

$$\exp\big(d(H_1, H_1\cap H_2) + d(H_2, H_1\cap H_2)\big)$$

$$= \frac{\sqrt{|H_1||H_2|}}{|H_1\cap H_2|} = \exp\big(d(H_1, H_2)\big)$$

(using part a))

6 a) Observe $|A+(A-2A)| < 2|A|$.

By Ruzsa covering, can find $X \subset G$, $|X| < 2$ s.t. $A-2A \subset A-A+X$.

So $X = \{x_0\}$, $A-2A \subset A-A+x_0$.

b) Suffices to prove $2(A-A) = 2A - 2A \subset A-A$.

But we have $2A - A \subset A - A - x_0$,

so $2A - 2A = A + (A-2A) \subset A + (A - A + x_0)$

$= (A + A - A) + x_0$

$\subset A - A - x_0 + x_0 = A - A$. ✓.

Note $0 \in A-A$

If $x = a_1 - a_2 \in A-A$, then $-x = a_2 - a_1 \in A-A$.